Let $f(x) = \dfrac{x^2 - 1}{x(x+4)^2}$.

SCORE: _____ / 20 PTS

[a] Find all discontinuities of f.

$x(x+4)^2 = 0$

$x = 0, -4$

[b] Find the limit of f at each discontinuity.

Each limit should be a number, ∞ or $-\infty$. **Write DNE only if the other possibilities do not apply.**

$\lim\limits_{x \to 0^+} \dfrac{x^2-1}{x(x+4)^2} = -\infty$

$\dfrac{-1}{0^+(16)}$

$\lim\limits_{x \to 0^-} \dfrac{x^2-1}{x(x+4)^2} = \infty$

$\dfrac{-1}{0^-(16)}$

$\lim\limits_{x \to 0} \dfrac{x^2-1}{x(x+4)^2}$ DNE

$\lim\limits_{x \to -4^+} \dfrac{x^2-1}{x(x+4)^2} = -\infty$

$\dfrac{15}{-4(0^+)}$

$\lim\limits_{x \to -4^-} \dfrac{x^2-1}{x(x+4)^2} = -\infty$

$\dfrac{15}{-4(0^+)}$

$\lim\limits_{x \to -4} \dfrac{x^2-1}{x(x+4)^2} = -\infty$

State the formal definition of "horizontal asymptote". **SCORE: _____ / 5 PTS**

f HAS A HORIZONTAL ASYMPTOTE AT $y=b$ IFF

$$\lim_{x\to\infty} f(x) = b \text{ OR } \lim_{x\to-\infty} f(x) = b$$

Find the equation(s) of all horizontal asymptote(s) of $f(x) = \dfrac{9-6e^{-x}}{13e^{-x}+4}$. **SCORE: _____ / 15 PTS**

$$\lim_{x\to\infty} \frac{9-6e^{-x}}{13e^{-x}+4} = \frac{9-0}{0+4} = \frac{9}{4}$$

$$\lim_{x\to-\infty} \frac{9-6e^{-x}}{13e^{-x}+4} = \lim_{x\to-\infty} \frac{9e^{x}-6}{13+4e^{x}} = \frac{0-6}{13+0} = -\frac{6}{13}$$

$$\frac{-\infty}{\infty}$$

$$y = \frac{9}{4} \text{ AND } y = -\frac{6}{13}$$

Let $f(x) = 7 - 3x^2 - x^3$. **SCORE: _____ / 30 PTS**

[a] Find $f'(x)$. **No credit will be given for using differentiation shortcuts from chapter 3.**

$$f'(x) = \lim_{h \to 0} \frac{7 - 3(x+h)^2 - (x+h)^3 - (7 - 3x^2 - x^3)}{h}$$

$$= \lim_{h \to 0} \frac{7 - 3(x^2 + 2xh + h^2) - (x^3 + 3x^2h + 3xh^2 + h^3) - 7 + 3x^2 + x^3}{h}$$

$$= \lim_{h \to 0} \frac{-6xh - 3h^2 - 3x^2h - 3xh^2 - h^3}{h}$$

$$= \lim_{h \to 0} (-6x - 3h - 3x^2 - 3xh - h^2)$$

$$= -6x - 3x^2$$

[b] Find the equation of the tangent to the graph of f at the point where $x = -1$.

$f(-1) = 7 - 3 + 1 = 5$ $\quad f'(-1) = 6 - 3 = 3$

$y - 5 = 3(x + 1)$

State the Squeeze Theorem. **SCORE: _____ / 10 PTS**

IF $f(x) \leq g(x) \leq h(x)$ ON AN OPEN INTERVAL AROUND $x=a$
(EXCEPT POSSIBLY AT $x=a$)

AND $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

THEN $\lim_{x \to a} g(x) = L$

Use the Squeeze Theorem to prove that $\lim_{x \to \infty} \frac{\sin x}{x} = 0$. **SCORE: _____ / 10 PTS**

$$-1 \le \sin x \le 1$$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x} \quad \text{AS } x \to \infty$$

$$\lim_{x \to \infty} -\frac{1}{x} = 0 = \lim_{x \to \infty} \frac{1}{x}$$

BY SQUEEZE THEOREM, $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

At time t hours, the position of an object moving in a straight line is $p(t) = \dfrac{38+5t}{4+t}$ meters. **SCORE: _____ / 25 PTS**

[a] What is the average velocity of the object from time $t = 2$ to time $t = 5$? **<u>Give the units of your answer.</u>**

$$\frac{p(5)-p(2)}{5-2} = \frac{\frac{63}{9}-\frac{48}{6}}{3} = \frac{7-8}{3} = -\frac{1}{3}\ \text{m/hr}$$

[b] What is the instantaneous velocity of the object at time $t = 2$? **<u>Give the units of your answer.</u>**

$$\lim_{b\to 2} \frac{\frac{38+5b}{4+b}-8}{b-2}$$

$$= \lim_{b\to 2} \frac{38+5b-8(4+b)}{(b-2)(4+b)}$$

$$= \lim_{b\to 2} \frac{6-3b}{(b-2)(4+b)} \xrightarrow{} -3$$

$$= \frac{-3}{6}$$

$$= -\frac{1}{2}\ \text{m/hr}$$

OR

$$\lim_{h\to 0} \frac{\frac{38+5(2+h)}{4+(2+h)}-8}{h}$$

$$= \lim_{h\to 0} \frac{\frac{48+5h}{6+h}-8}{h}$$

$$= \lim_{h\to 0} \frac{48+5h-8(6+h)}{h(6+h)}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}(6+h)}$$

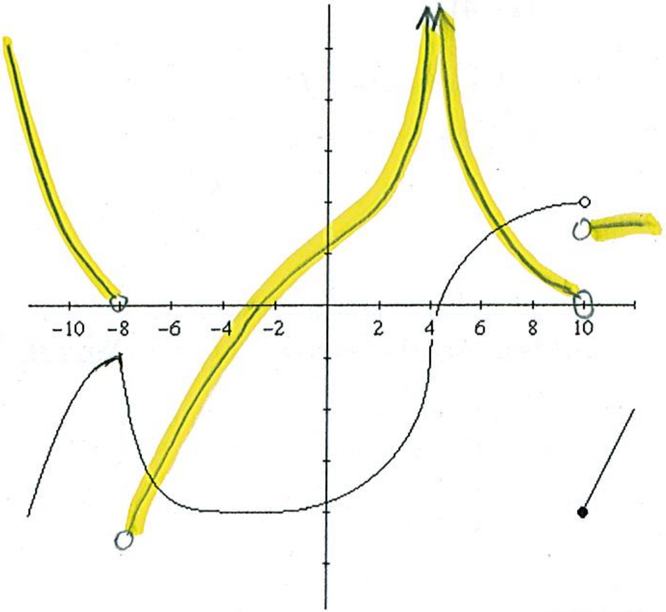
$$= \frac{-3}{6}$$

$$= -\frac{1}{2}\ \text{m/hr}$$

The graph of $f(x)$ is shown on the right.

SCORE: ___ / 20 POINTS

[a] List all $x -$ values for which $f'(x)$ does not exist.
For each $x -$ value, give a **brief** reason why not.



x = -8 (CUSP)

x = 4 (VERTICAL T.L.)

x = 10 (DISCONTINUITY)

[b] Sketch a graph of $f'(x)$ **on the same axes**.

The monthly heating/cooling cost for the classrooms depends on the temperature at which they are kept. **SCORE: _____ / 15 PTS**
Let $c = g(t)$, where c is the monthly cost (in hundreds of dollars), and t is the temperature (in $^\circ F$).

[a] What are the units of $g'(t)$?

HUNDREDS OF \$/°F

[b] What does $g'(62) = -5$ mean ? **<u>Give the units for all numbers in your answer.</u>**

IF THE ROOMS ARE KEPT AT 62°F,
THEN FOR EACH 1°F HIGHER THE ROOMS ARE KEPT,
\$500 WOULD BE SAVED EACH MONTH

[c] Is there a value of t_0 for which you would expect $g'(t_0) > 0$? Why or why not ?

YES. IF THE ROOMS ARE KEPT EXTREMELY HOT (EG. >100°F)
RAISING THE TEMPERATURE WOULD COST MORE